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МОСКОВСКИЙ ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ
имени М.В.ЛОМОНОСОВА

Вариант 10 класс

Место проведения Москва
город

ПИСЬМЕННАЯ РАБОТА

Олимпиада школьников „Ломоносов"
наменование олимпиады

по математике
профиль олимпиады

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фамилия, имя, отчество участника (в родительном падеже)

+1 [signature]

Дата

«13» апреля 2025 года

Подпись участника

95-49-32-96
(160.8)

50 (пятьдесят)

$$\sqrt{\log_2^2 x + 3\log_2 x - \sqrt{\log_2^2 x + 3\log_2 x - 4}} \geq \log_2 x + 1$$

$$\sqrt{t^2 + 3t - \sqrt{(t+4)(t-1)}} \geq t + 1$$

$t + 1 \geq 0$

$y \geq 0$

$(t+4)(t-1) \geq 0$

$\sqrt{y^2 - y + 4}$

$y = \sqrt{(t+4)(t-1)}$

$y^2 - y$

$$\cancel{t^2} + 3t - \cancel{4} - \sqrt{(t+4)(t-1)} + \cancel{4} \geq \cancel{t^2} + 2t + 1$$

$$t - 1 \geq \sqrt{(t+4)(t-1)}$$

$t - 1 \geq 0$

$\boxed{t \geq 1}$

$t^2 - 2t + 1 \geq (t+4)(t-1)$

$(t-1)(t-1-t-4) \geq 0$

$-5(t-1) \geq 0$

$t - 1 \leq 0$

$t \leq 1$

$t \geq 1$

$t = 1$

$x = 2$

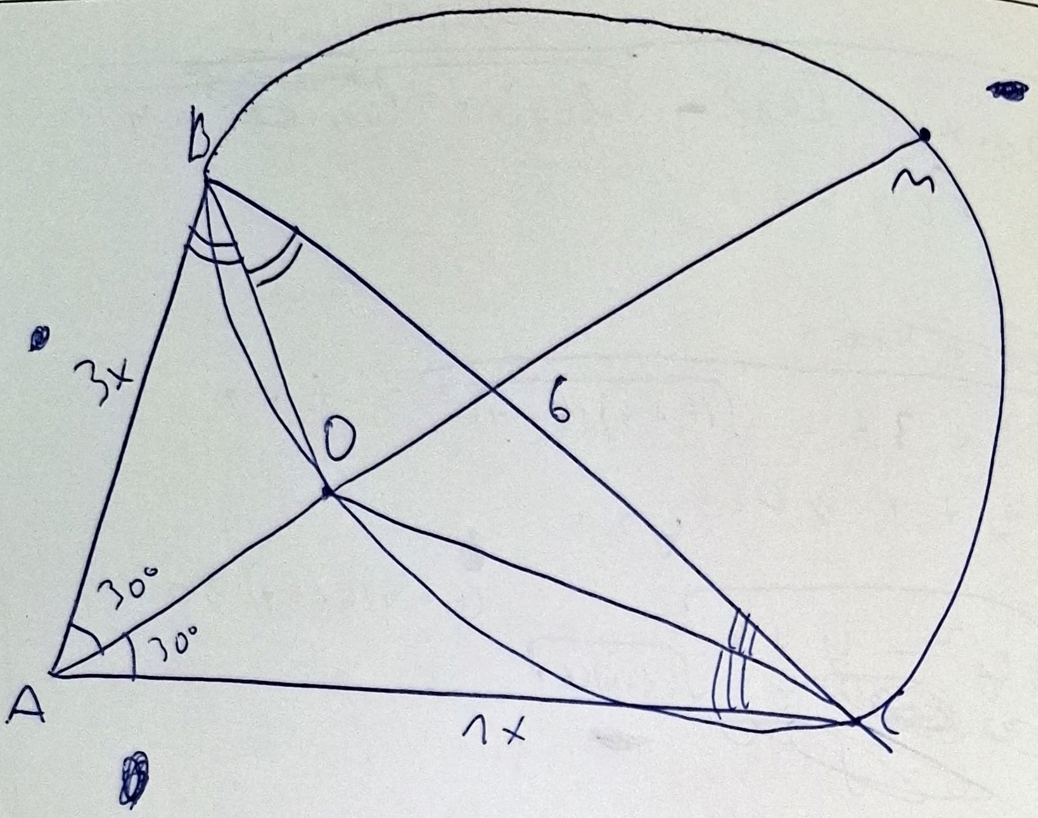
$\log_2 x = 1 \Leftrightarrow 2^1 = x$

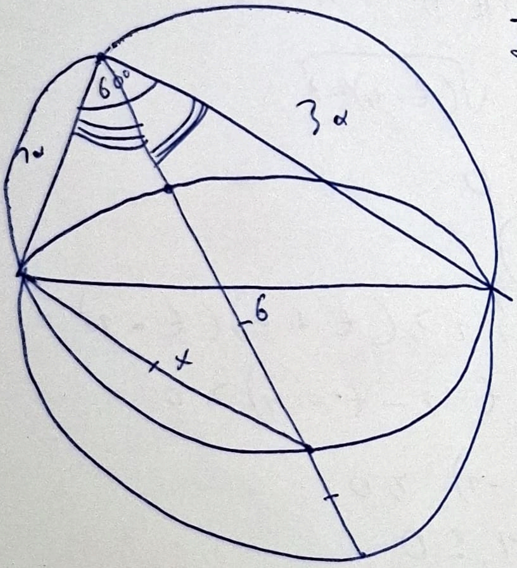
$f(x_0) = 16x_0$

$f(2) = 32.$

Расп-

Суд





$\frac{6}{\sin 60^\circ} = \frac{x}{\sin 30^\circ}$

$x = \frac{1}{2} \cdot \frac{6}{\frac{\sqrt{3}}{2}} =$

$= \frac{1}{2} \cdot \frac{12}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

$2 \cdot \sqrt{3} \cdot 2 =$

$= 4\sqrt{3}$

~~$OM = 2 \cdot x = 4\sqrt{3}$~~

~~$2x = 4\sqrt{3}$~~

$4\sqrt{3}$

т.к. $OM = 2x$ – радиус, то

$OM = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$

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$\sqrt{\sqrt{\cos x} + 2\sqrt{-3\cdot\sin x}} > 2\cdot\sqrt{\sqrt{\cos x} - \sqrt{-3\cdot\sin x}}$

$2\cdot\sqrt{\sqrt{\cos x} - \sqrt{-3\cdot\sin x}}$

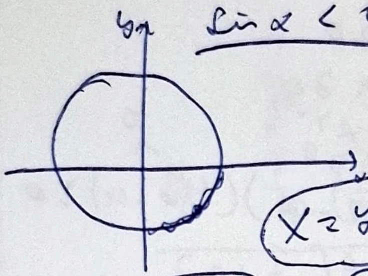
$\sqrt{\sqrt{\cos x} + 2\cdot\sqrt{-3\cdot\sin x}}$

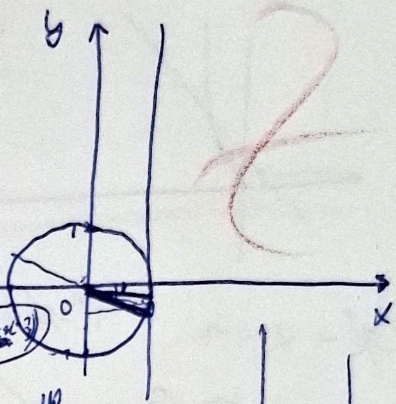
$> 2\cdot\sqrt{\sqrt{\cos x} - \sqrt{-3\cdot\sin x}}$

$\cos x > 0$

$-3\cdot\sin x > 0$

$\sin x < 0$





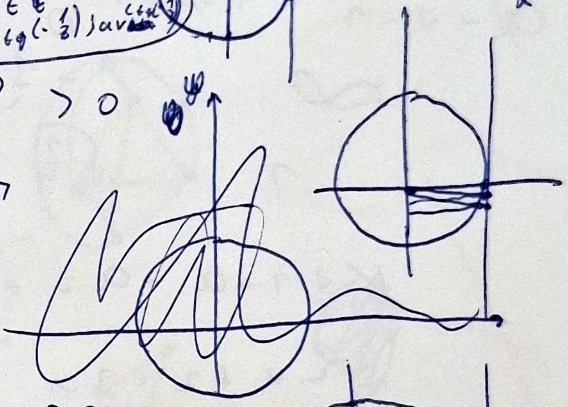
$x = y + 2\pi k,\ k\in\mathbb{Z}$

$y \in (\operatorname{arctg}(-\tfrac{1}{3}); \operatorname{arctg}(\dots))$

$\sqrt{\cos x} - \sqrt{-3\cdot\sin x} > 0$

$\sqrt{\cos x} > \sqrt{-3\cdot\sin x}$

$\Updownarrow$

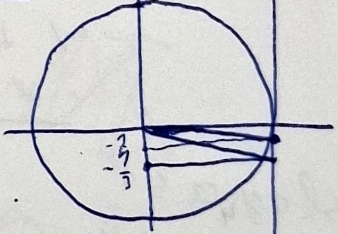


$\cos x > -3\cdot\sin x$

$\cos x + 3\cdot\sin x > 0$

$1 + 3\cdot\operatorname{tg} x > 0$

$\operatorname{tg} x > -\frac{1}{3}$



$\sqrt{\cos x} + 2\cdot\sqrt{-3\cdot\sin x} > 4\cdot\sqrt{\cos x} - 4\cdot\sqrt{-3\cdot\sin x}$

$\operatorname{arctg}(-\tfrac{1}{6})$

$6\sqrt{-3\cdot\sin x} > 3\cdot\sqrt{\cos x}$

$2\cdot\sqrt{-3\cdot\sin x} > \sqrt{\cos x}$

$\Updownarrow$

$-6\cdot\sin x > \cos x$

$0 > \cos x + 6\cdot\sin x$

$0 > 1 + 6\operatorname{tg} x$

$\operatorname{tg} x < -\frac{1}{6}$

y и $-x \geqslant \operatorname{arctg}(-\tfrac{1}{3})$

$\operatorname{arctg}(-\tfrac{1}{3})$

$\geqslant \dots \operatorname{arctg}(-\tfrac{1}{6})$

$-\frac{1}{3} < \operatorname{tg} x < -\frac{1}{6}$

$|x+1-a| + |4^x - a| = 4^x - x - 1$

$4^x - a < 0$
$x+1-a<0$
$\frac{1}{4} \le a \le 4$

1) $-$ $-$
2) $+$ $+$
3) $+$ $-$
4) $-$ $+$

$a \in [0; 2)$

$x \; a < 0$

$a < 0$

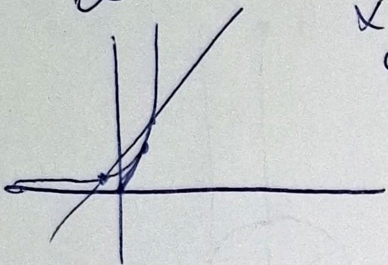
$[-1; 1]$

$\cancel{x+1-x-a} > 0$

$x+1-a<0$
$4^x - a > 0$

∞

$x+1-a \ge 0$
$4^x - a \le 0$



$4^x - a \ge 0$
$x+1-a \ge 0$

$4^x = x+1$ $\quad x \ne -\frac{1}{4}$ $\quad x > 0$

$\cancel{a} - x - 1 + 4^x - a = 4^x - x - 1$

0

$-\frac{1}{2}$

$\begin{cases} (x+1-a)(4^x - a) \ge 0 \\ a \in [\frac{1}{4}; 2). \end{cases}$

∞

$\sqrt{\frac{1}{4}} = \frac{1}{2}$ $\quad \frac{1}{2}$

$4^x > a$ $\quad \log_4 a$

1

$x > 1$

$x+1-a-a = -x-1$

$2x+2 = 2a$
$x+1 = a$
$\underline{x = a-1}$.

$x \ge a-1$

$-1 \le \log_4 a \le 1$
$\underline{\frac{1}{4} \le a \le 4}$

$\underline{a-1 \in [-1; 1]}$.

$\underline{a \in [0; 2)}$.

$4^x - a < 0$
$x+1-a<0$

$a \le 4^x$
$a \le x+1$

$a - \cancel{x} - 1 - 4^x + a =$
$= 4^x - \cancel{x-1}$
$2a = 4^x \cdot 2$
$a = 4^x$
$x = \log_4 a$

$\frac{1}{4} \le 4^x \le 4$

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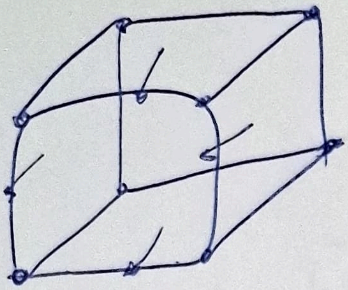
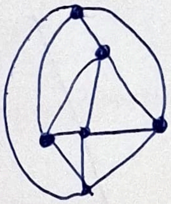
$|x+1-a| + |4^x - a| = 4^x - x - 1$

a



$\frac{1}{4}$

$$\frac{4 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^3}{A_i -}$$



$2^?$

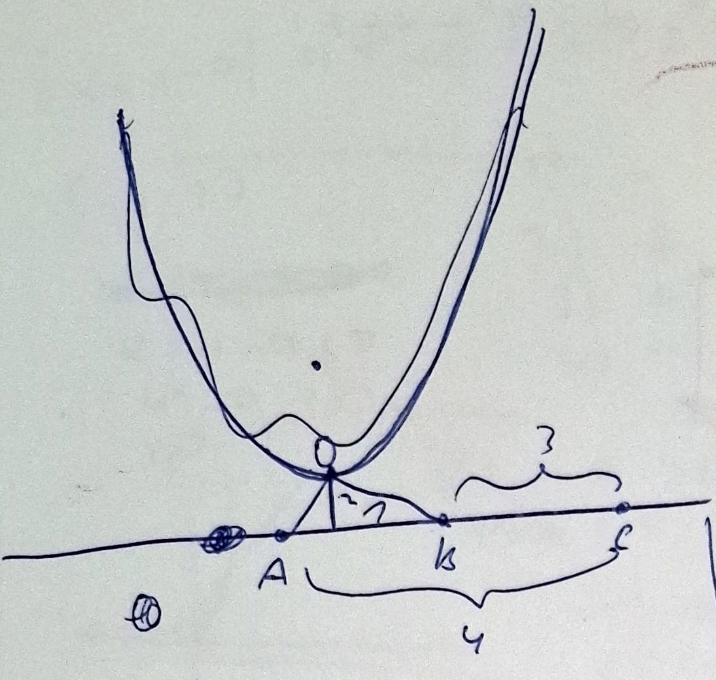
$2 \cdot \frac{1}{2} \cdot \frac{1}{2}$ $\qquad \frac{2}{8} = \frac{1}{4}$

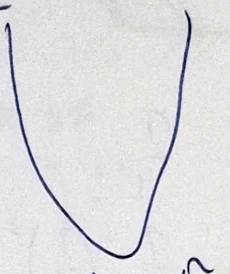
$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\frac{2}{16} = \frac{1}{8}$ $\qquad 3$ $\qquad \frac{3}{81} =$ $\qquad \frac{1}{27}$

$2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$





$y = x^2$

$M(x_0, y_0)$

$2 \cdot 1 \cdot \frac{1}{2} = \underline{1}$

$y = kx + b$

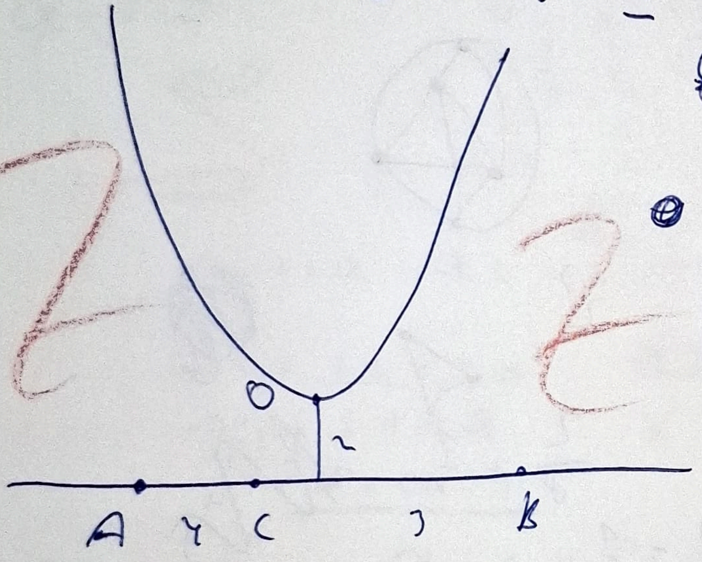
$x^2 - kx - b = 0$

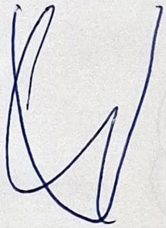
$k^2 + 4b = 0$

$y_0 = kx_0 + b$

$b = -\frac{k^2}{4}$

$y_0 = kx_0 - \frac{k^2}{4}$





$\frac{k^2}{4} - kx_0 + y_0 = 0$

$k^2 - 4kx_0 + y_0 4 = 0$

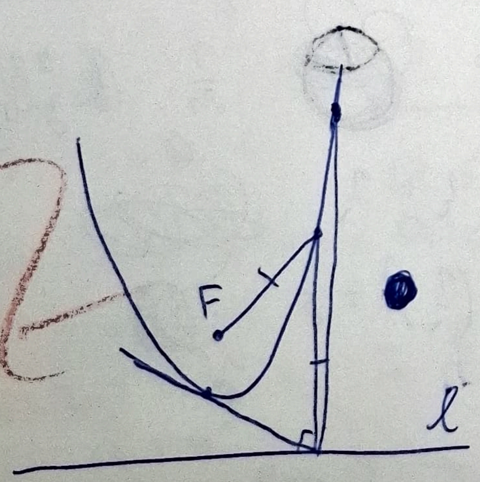
$b = -\frac{k^2}{4}$

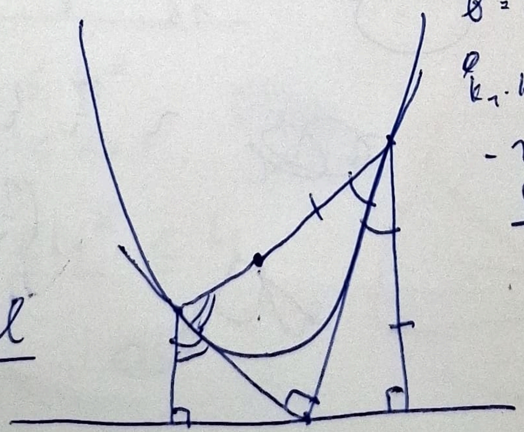
$k_1 \cdot k_2 = 4y_0$

$-1 = 4y_0$

$\underline{y_0 = -\frac{1}{4}}$

$2 \cdot 7 \cdot \frac{1}{2} = \underline{7}$





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417840

99

100

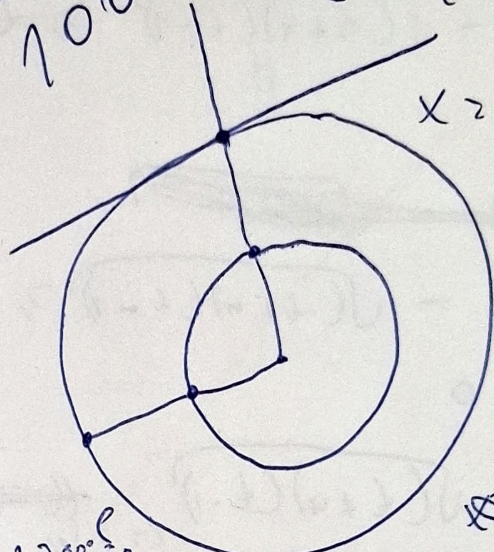
$$r = R \cdot \frac{180^\circ - \alpha^\circ \pi}{180^\circ}$$

$$\cancel{2\pi R} \cdot \frac{180^\circ - \alpha^\circ \cdot \pi}{180^\circ} \cdot \frac{360^\circ - \alpha^\circ}{360^\circ} = \frac{\alpha^\circ}{360^\circ} \cdot \cancel{2\pi R}$$

$$\alpha^\circ = \left(1 - \frac{\alpha^\circ \pi}{180^\circ}\right) \cdot (360^\circ - \alpha)$$

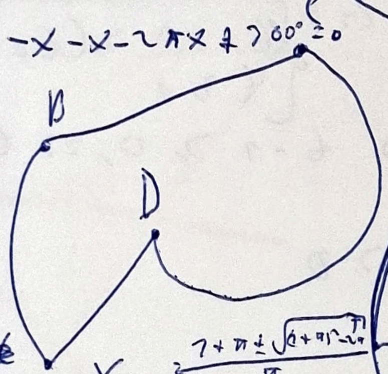
$$x = \left(1 - x \cdot \frac{\pi}{180^\circ}\right)(360 - x)$$

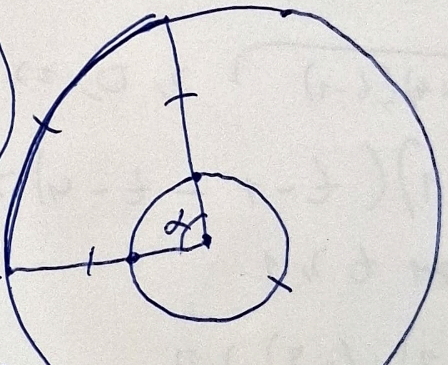
1849



$$x = 360^\circ - 2\pi x - x + x^2 \cdot \frac{\pi}{180^\circ}$$

$$x^2 \cdot \frac{\pi}{180^\circ} - x - x - 2\pi x + 360^\circ = 0$$





$$x_{3,4} = \frac{1 + \pi \pm \sqrt{(1+\pi)^2 - 2\pi}}{\frac{\pi}{180^\circ}}$$

$$x^2 \cdot \frac{\pi}{180^\circ} - 2x(1+\pi) + 360^\circ = 0$$

$$2\pi r - \frac{\alpha^\circ}{360^\circ} \cdot 2\pi r$$

$$2\pi r \cdot \left(\frac{360^\circ - \alpha^\circ}{360^\circ}\right)$$

$$r = R - \frac{\alpha^\circ}{180^\circ} \cdot \pi \cdot R$$

$$r = R\left(1 - \frac{\alpha^\circ}{180^\circ} \cdot \pi\right)$$

$$x = \frac{1 + \pi + \sqrt{1 + \pi^2}}{\frac{\pi}{180^\circ}}$$

$$\frac{\alpha^\circ}{360^\circ} \cdot 2\pi \cdot R$$

$$\frac{\pi}{180} \cdot \alpha^\circ = \alpha_r$$

$$\overline{1 + \pi + \sqrt{1 + \pi^2}}.$$

№ 1.

$$\sqrt{\log_2^2 x + 3\cdot\log_2 x - \sqrt{\log_2^2 x + 3\log_2 x - 4}} \geq$$

$$\geq \log_2 x + 1$$

$\underline{x > 0}$, $\underline{\log_2 x + 1 \geq 0}$, $\log_2^2 x + 3\log_2 x - 4 \geq 0$,

$\log_2^2 x + 3\log_2 x - \sqrt{\log_2^2 x + 3\log_2 x - 4} \geq 0.$

$t = \log_2 x$

$$\sqrt{t^2 + 3t - \sqrt{(t+4)(t-1)}} \geq t + 1$$

$$\Updownarrow$$

$$\begin{cases} t^2 + 3t - \sqrt{(t+4)(t-1)} \geq t^2 + 2t + 1 \\ t + 1 \geq 0 \end{cases}$$

$$\begin{cases} t - 1 \geq \sqrt{(t+4)(t-1)} \\ t + 1 \geq 0 \end{cases} \Leftrightarrow \begin{cases} (t-1)^2 \geq (t+4)(t-1) \\ t \geq 1 \end{cases}$$

$\sqrt{(t+4)(t-1)} \geq 0, \Rightarrow t - 1 \geq 0, \Rightarrow t \geq 1$

$$\begin{cases} (t-1)(t-1-t-4) \geq 0 \\ t \geq 1 \end{cases}$$

$$\begin{cases} (t-1)\cdot(-5) \geq 0 \\ t \leq 1 \end{cases}$$

$0 \leq t - 1 \leq 0,$

$1 \leq t \leq 1, \Rightarrow \underline{t = 1}$ – необход. усл.

Докажем, что $t = 1$ подходит

$(!)\sqrt{1^2 + 3\cdot 1 - \sqrt{5\cdot 0}} \geq 1 + 1$

(!) $\sqrt{4} \geq 2$

$2 \geq 2$ – верно, $\Rightarrow$ $t = 1$ – решение

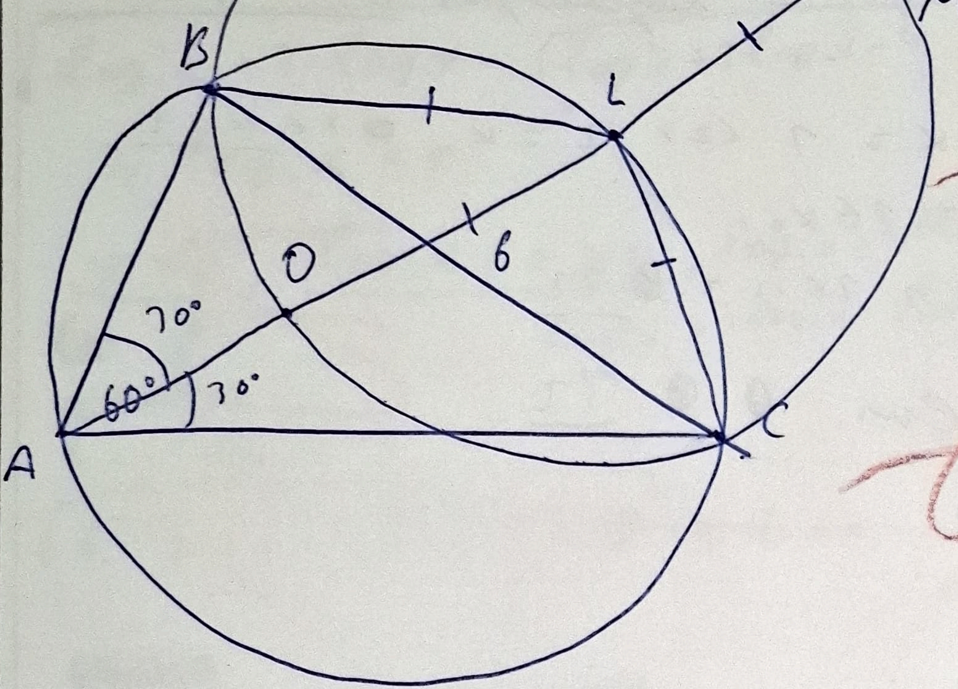
$\log_2 x = 1 \Leftrightarrow 2^1 = x$, т.е. $x = 2$

$f(x_0) = 16 x_0$

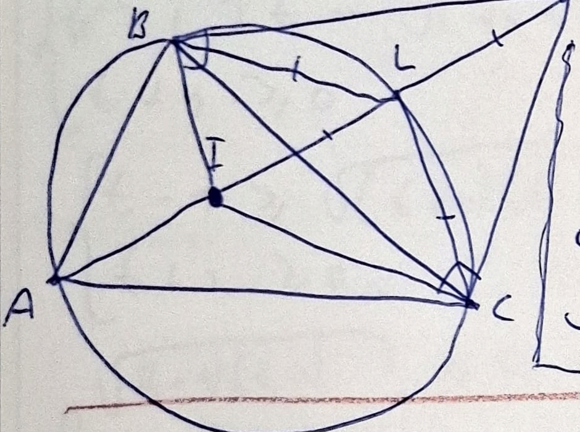
$f(2) = 16 \cdot 2 = 32.$

Ответ. 32.

№2.



Лемма о трезубце:



I_a | т. I – центр вписанной окр. $\triangle ABC$,
$AI \cap (ABC) =$ т. L.
т. I_a – центр вневписанной окр. $\triangle ABC$ со кас. ст. BC.
Тогда $\underline{LB = LC = IL = LI_a}$.

Пусть $AO \cap (ABC)$ в т. L. Тогда по лемме о трезубце $LB = LC = LO = LM$,
$\underline{OM} = 2OL = \underline{2LC}$

По теореме синусов для $\triangle ABC$ и $\triangle ALC$:

$\frac{BC}{\sin 60°} = \frac{LC}{\sin 30°}$ ~~L ∈ (AB~~ т. A, B, C и L ∈ 1 окр

$LC = \frac{6}{\sin 60^\circ} \cdot$ [redacted] $\sin 30^\circ$

$LC = \frac{6}{\frac{\sqrt{3}}{2}} \cdot \frac{1}{2} = \frac{6}{\sqrt{3}} = \underline{2\sqrt{3}}.$

$\underline{OM} = 2LC = \underline{4\sqrt{3}}.$

Ответ: ~~OM = 4~~ ~~4~~ ~~4~~ $\underline{4\sqrt{3}}$.

№ 3.

$$\sqrt{\sqrt{\cos x} + 2\cdot\underbrace{\sqrt{-3\cdot\sin x}}_{\geq 0}} > 2\cdot\sqrt{\sqrt{\cos x} - \sqrt{-3\cdot\sin x}}$$

$\cos x \geq 0$, $\sin x \leq 0$.

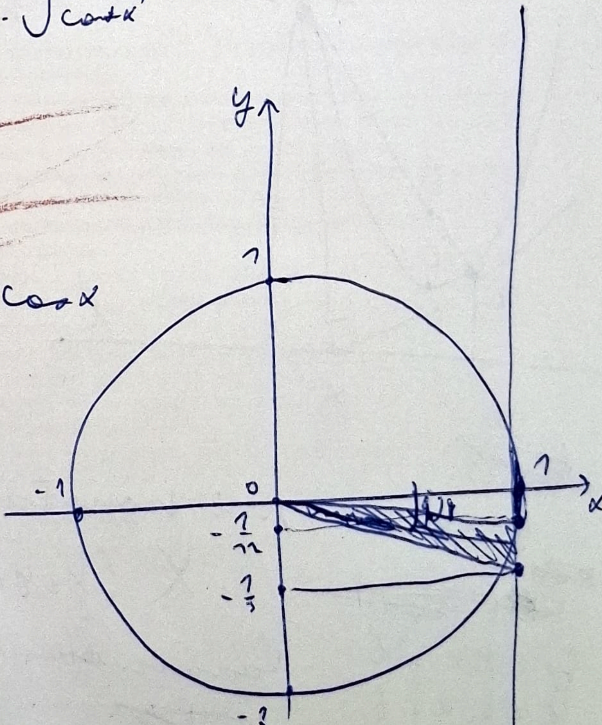
$\sqrt{\cos x} - \sqrt{-3\cdot\sin x} \geq 0 \iff \begin{cases} \cos x \geq -3\cdot\sin x \\ \cos x \geq 0 \\ \sin x \leq 0 \end{cases} \iff \begin{cases} \operatorname{tg} x \geq -\frac{1}{3} \\ \cos x \geq 0 \\ \sin x \leq 0 \end{cases}$

$$\begin{cases} \sqrt{\cos\alpha} + 2\cdot\sqrt{-3\cdot\sin x} > 4\cdot\sqrt{\cos x} - 4\cdot\sqrt{-3\sin x} \\ \cos\alpha \geq 0 \\ \sin x \leq 0 \\ \operatorname{tg}\alpha \geq -\frac{1}{3} \end{cases}$$

$$\begin{cases} 6\cdot\sqrt{-3\cdot\sin x} > 3\cdot\sqrt{\cos x} \\ \cos\alpha \geq 0 \\ \sin x \leq 0 \\ \operatorname{tg}\alpha \geq -\frac{1}{3} \end{cases}$$

$$\begin{cases} -12\cdot\sin x > \cos x \\ \cos\alpha \geq 0 \\ \sin\alpha \leq 0 \\ \operatorname{tg} x \geq -\frac{1}{3} \end{cases}$$

$$\begin{cases} \operatorname{tg}\alpha < -\frac{1}{12} \\ \cos\alpha \geq 0 \\ \sin\alpha \leq 0 \\ \operatorname{tg}\alpha \geq -\frac{1}{3} \end{cases}$$

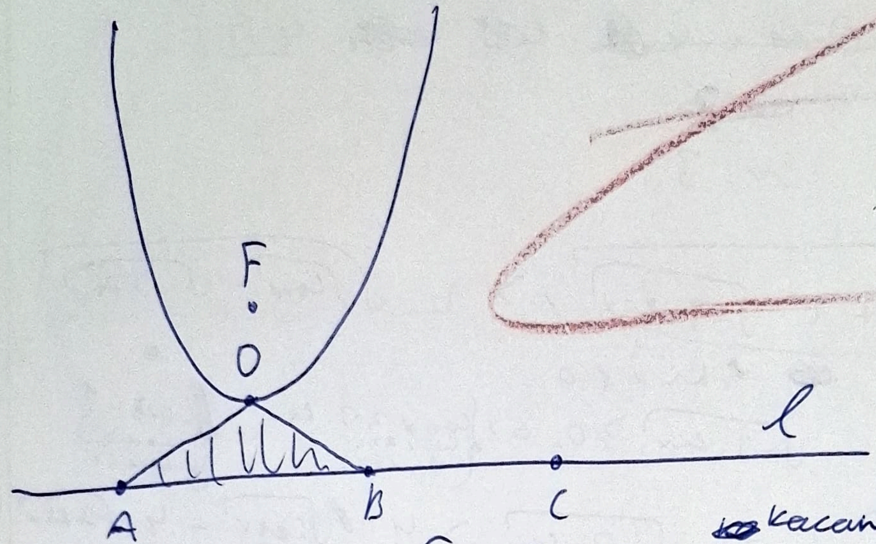


$\alpha \in \left(\operatorname{arctg}\left(-\frac{1}{12}\right); \operatorname{arctg}\left(-\frac{1}{3}\right)\right)$

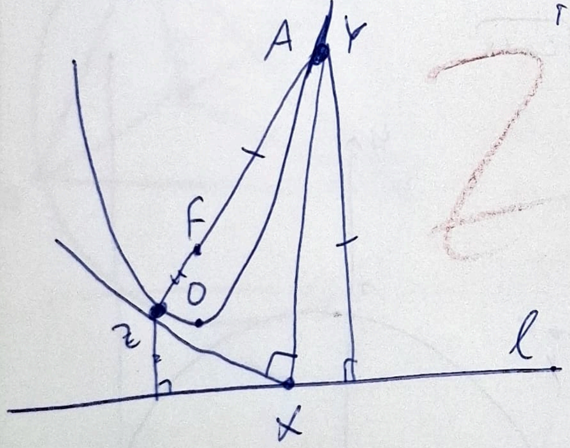
т.к. $\cos\alpha \geq 0$ и $\sin x \leq 0$, то $x = -\alpha + 2\pi k$, $k \in \mathbb{Z}$.

Ответ $x = -\alpha + 2\pi k,\ k \in \mathbb{Z}$
$\alpha \in (\operatorname{arcctg}(-\frac{1}{12}); \operatorname{arcctg}(-\frac{1}{3}))$.

№ 6.



Докажем, что ГМТ A: ~~касат~~ касат к параболе перпендикулярны – это директриса параболы, F – фокус.
т.е. прямая. Пусть l – директриса,
1) Докажем, что $\forall$ ~~касат~~ т. $X \in l$: касат перпен~~дикул~~



$y = x^2$ – ур.е параболы
т. $X(x_0 y_0)$

$y = k_1 x + b$ – касательные
$y = k_2 x + b$
$\underline{k_1 \cdot k_2 = -1}$.

$y = kx + b$
$x^2 - kx - b = 0$
$D = 0 \quad k^2 + 4b = 0$
$b = -\frac{k^2}{4}$

$y_0 = k x_0 + b$
$y_0 = k x_0 - \frac{k^2}{4}$

$y_0 = kx_0 - \frac{k^2}{4}$

$\frac{k^2}{4} - kx_0 + y_0 = 0$

$k^2 - 4kx_0 + 4y_0 = 0$

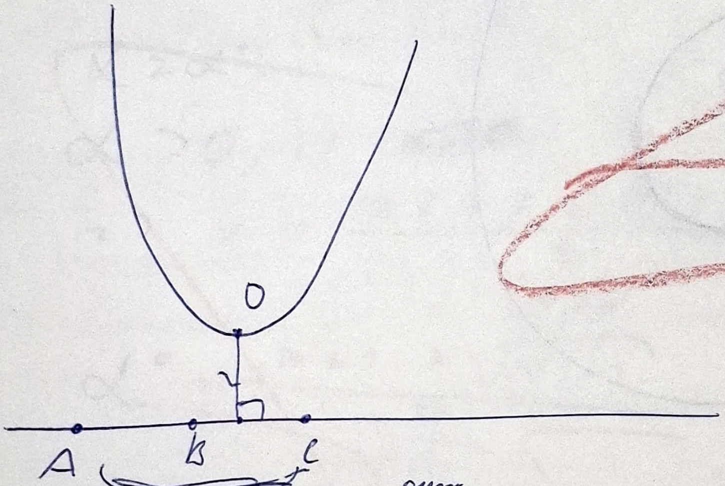
по т. Виета:

$k_1 \cdot k_2 = 4y_0$

$k_1 \cdot k_2 = -1$

$\underline{y_0 = -\frac{1}{4}}$, т.е. все т. X лежат на пр. $\underline{y = -\frac{1}{4}}$.

Значит, т. A, B, C $\in$ 1 пр.



1) $AC = 4$ т. $B \in$ отр AC

$BC = 3$

~~A AB~~

$AB = 1$

$\underline{S(ABO) = 2 \cdot 1 \cdot \frac{1}{2} = 1}$

2) $AC = 4$

$BC = 3$

т. $C \in$ отр AB

$\underline{AB = 7}$.

$\underline{S(ABO)} = 7 \cdot 2 \cdot \frac{1}{2} = \underline{7}$

Ответ $\underline{1 \text{ или } 7}$.

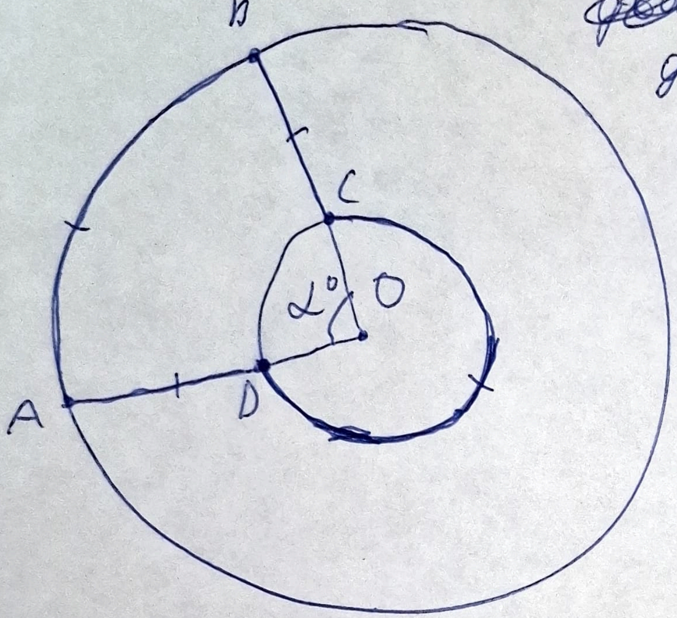
№ 8.

~~AB~~ $\breve{AB}$ – меньшая часть

$\breve{CD}$ – большая часть, то

то рисунок такой:

~~две~~ две окр. с центром в т. O



$\angle AOB = \alpha^\circ$.

$OA = R$

$OC = r$

$|\breve{AB}| = \frac{\alpha^\circ}{360^\circ} \cdot 2\pi R$

$|BC| = |\breve{AB}|$

$r = OC = BO - BC = R - \frac{\alpha^\circ}{360^\circ} \cdot 2\pi R =$

$= R\left(1 - \frac{\alpha^\circ \pi}{180^\circ}\right)$

$|\breve{CD}| = 2\pi \cdot r \cdot \left(\frac{360^\circ - \alpha^\circ}{360^\circ}\right) =$

$= 2\pi \cdot R\left(1 - \frac{\alpha^\circ \pi}{180^\circ}\right)\left(\frac{360^\circ - \alpha^\circ}{360^\circ}\right) =$

$= |\breve{AB}| = \frac{\alpha^\circ}{360^\circ} \cdot 2\pi R$

Ответ $x = -\alpha + 2\pi k, \; k \in \mathbb{Z}$

$1, \; (\operatorname{arcctg}(-\frac{1}{2}); \operatorname{arcctg}(-\frac{1}{3}))$.

$$\alpha^\circ = \left(1 - \frac{\alpha^\circ \pi}{180^\circ}\right)\left(360^\circ - \alpha^\circ\right)$$

$$\alpha^\circ = 360^\circ - \alpha^\circ + 2\pi\alpha^\circ + (\alpha^\circ)^2 \frac{\pi}{180^\circ}$$

$$\frac{\pi}{180^\circ} x^2 - 2x(\pi + 1) + 360^\circ = 0$$

$$x_{1,2} = \frac{\pi + 1 \pm \sqrt{(\pi+1)^2 - 2\pi}}{\frac{\pi}{180^\circ}}$$

$x = \alpha^\circ$

$\alpha > 0, \Rightarrow$

$$\Rightarrow x = \frac{\pi + 1 + \sqrt{\pi^2 + 1}}{\frac{\pi}{180^\circ}}$$

$$\alpha^\circ = \frac{\pi + 1 + \sqrt{\pi^2 + 1}}{\frac{\pi}{180^\circ}}$$

$$\alpha_{\text{рад.}} = \alpha^\circ \cdot \frac{\pi}{180^\circ}$$

$$\alpha_{\text{рад}} = \pi + 1 + \sqrt{\pi^2 + 1}.$$

№4

$|x+1-a| + |4^x - a| = 4^x - x - 1.$

1) $|x+1-a| < 0$

$|4^x - a| < 0$

$a - \cancel{x} - \cancel{1} + a - 4^x = 4^x - \cancel{x} - \cancel{1}$

$4^x = a$

$x = \log_4 a$ — 1 реш

$-1 \le \log_4 a \le 1$

$\begin{cases} \frac{1}{4} \le a \le 4 \\ a > x+1 \\ a > 4^x \end{cases}$

$\begin{cases} (x+1-a)(4^x - a) \ge 0 \\ a \in (\frac{1}{4}; 2]. \end{cases}$

2) $\begin{cases} a < x+1 \\ a < 4^x \end{cases}$

$\begin{cases} a < x+1 \\ a < 4^x \\ a \in (0; 2). \end{cases}$

~~$x+1 + a + 4^x$~~

$x + 1 + \cancel{4^x} - 2a = \cancel{4^x} - x - 1$

$a = x+1$

$x = a - 1$

$-1 \le a - 1 \le 1$

$0 \le a \le 2$

$a \in [0; 2].$

3) + −

4) − +

$x + 1 - \cancel{a} - 4^x + \cancel{a} = 4^x - x - 1$

$4^x = x+1$

$x = 0$

$x = \cdot 1$ — корни

X

4) $4^x - \cancel{a} + a - x + 1 = 4^x - x + 1$

∞ x